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Nuclear forces in a Kerr–Newman background space

D J Rowan
Department of Mathematics, Imperial College, London SW7, UK

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Abstract. Asymptotic methods are used to solve the radial equation of a static scalar meson field in a Kerr–Newman background space. The solutions thus derived are used to obtain the form of the nuclear potential in the presence of a large mass, small charge, slowly rotating Kerr–Newman black hole, and to show that the meson field vanishes as the source approaches the event horizon.

1. Introduction

In a recent paper (Rowan and Stephenson 1976, to be referred to as I) the scalar meson field of a baryon in a Schwarzschild background space has been examined. Using a modified Liouville–Green technique, an approximate form for the nuclear potential was derived and shown to vanish as the baryon approached the event horizon. This was in agreement with an earlier paper by Teitelboim (1972), and also with the 'no hair' theorem, but differed from Teitelboim's work by obtaining an explicit form for the potential.

Similar work has been carried out for the corresponding electrostatic problem of a point charge in a Schwarzschild background space (see Cohen and Wald 1971, and Hanni and Ruffini 1973), and more recently in a Kerr background space with the charge located on the axis of symmetry (see Cohen *et al* 1974). In this paper the nuclear potential of a baryon on the axis of symmetry of a Kerr–Newman black hole is derived using a similar approximation to that used in I. For the approximation to be valid the black hole must have large mass, small charge, and be slowly rotating.

2. Basic equations

As in I we start with the generally covariant equation

$$(\Box^2+\mu^2)\Phi=-g\int_{-\infty}^{\infty}\frac{\delta^{(4)}(x-x'(\lambda))}{\sqrt{-g_4}}\,\mathrm{d}s(\lambda),\qquad(2.1)$$

which corresponds to the massive scalar meson field of a baryon whose world line is $x'(\lambda)$, μ being the inverse Compton wavelength of the π-meson, and g the coupling constant.

Consider a baryon situated in the exterior region of a Kerr–Newman black hole, with metric in Boyer–Lindquist coordinates given by

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\,d\phi - a\,dt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\,d\theta^2, \tag{2.2}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$. Assuming that the baryon is at rest on the axis of symmetry at $r = b$, $\theta = 0$, we treat the quasi-static in-fall (a series of static problems) of the baryon down the axis of rotation so that Φ is independent of both ϕ and t. Thus combining (2.1) and (2.2) we have

$$\left[\frac{\partial}{\partial r}\left(\Delta\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \rho^2\mu^2\right]\Phi = g\frac{(\Delta - a^2\sin^2\theta)^{1/2}}{2\pi\rho}\delta(r-b)\delta(\cos\theta - 1), \tag{2.3}$$

where

$$\int \delta(r-b)\,dr = 1 \qquad \text{and} \qquad \int \delta(\cos\theta - 1)\sin\theta\,d\theta\,d\phi = 2\pi.$$

Writing

$$\Phi = \sum_{l=0}^{\infty} R_l(r)S_l(\theta) \tag{2.4}$$

and substituting this into (2.3) reduces the problem to a radial equation

$$\frac{d}{dr}\left(\Delta\frac{dR_l}{dr}\right) - (\mu^2 r^2 + \lambda_l)R_l = g\frac{\Delta^{1/2}S_l(0)}{(r^2+a^2)^{1/2}}\delta(r-b), \tag{2.5}$$

where the λ_l occurring in (2.5) are the eigenvalues for the prolate spheroidal harmonics given by

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_l}{d\theta}\right) + (\lambda_l - a^2\mu^2\cos^2\theta)S_l = 0 \tag{2.6}$$

and the normalisation of the spheroidal harmonics has been taken as

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)|S_l(\theta)|^2 = 1. \tag{2.7}$$

Putting $Mx = r - r_+$ and $2Md = r_+ - r_-$ into (2.5), where r_+ and r_- are the outer and inner horizons defined by

$$r_+ = M + [M^2 - (a^2+Q^2)]^{1/2} \qquad \text{and} \qquad r_- = M - [M^2 - (a^2+Q^2)]^{1/2} \tag{2.8}$$

leads to the equation

$$\frac{d}{dx}\left(x(x+2d)\frac{dR_l}{dx}\right) - [N^2(x+d+1)^2 + \lambda_l]R_l = g\frac{\Delta^{1/2}S_l(0)}{M[M^2(x+d+1)^2+a^2]^{1/2}}\delta(x - x_b), \tag{2.9}$$

where $N^2 = \mu^2 M^2$ and $x_b = (b - r_+)/M$.

3. Solution of the radial equation outside the source

Equation (2.9) can be written in normal form by writing $R_l(x) = Z_l(x)[x(x+2d)]^{-1/2}$

and, for $x \neq x_b$, becomes

$$\frac{d^2 Z_l}{dx^2} - \left(N^2 \frac{(x+d+1)^2}{x(x+2d)} + \frac{\lambda_l}{x(x+2d)} - \frac{d^2}{x^2(x+2d)^2}\right) Z_l = 0. \tag{3.1}$$

This equation does not appear to have exact solutions in terms of known functions. Consequently we generate approximate solutions in a manner similar to that in I. Firstly we change the independent variable x, in (3.1), to ξ by the transformation $x = x(\xi)$ and transform the dependent variable using $G_l(\xi) = (d\xi/dx)^{1/2} Z_l(x)$ giving the equation

$$\frac{d^2 G_l}{d\xi^2} = \left[\left(N^2 \frac{(x+d+1)^2}{x(x+2d)} + \frac{\lambda_l}{x(x+2d)} - \frac{d^2}{x^2(x+2d)^2}\right)\frac{1}{\xi'^2} + \frac{\xi'''}{2\xi'^3} - \frac{3\xi''^2}{4\xi'^4}\right] G_l, \tag{3.2}$$

where $\xi' = d\xi/dx$. Then writing $K^2 = N^2 + \lambda_l$, $\alpha^2 = \lambda_l/K^2$ and $\beta^2 = N^2/K^2$, (3.2) becomes

$$\frac{d^2 G_l}{d\xi^2} = \left\{\left[K^2\left(\beta^2 \frac{(x+d+1)^2}{x(x+2d)} + \frac{\alpha^2}{x(x+2d)}\right) - \frac{d^2}{x^2(x+2d)^2}\right]\frac{1}{\xi'^2} + \frac{\xi'''}{2\xi'^3} - \frac{3\xi''}{4\xi'^4}\right\} G_l. \tag{3.3}$$

Now choosing

$$\xi'^2 = \beta^2 \frac{(x+d+1)^2}{x(x+2d)} + \frac{\alpha^2}{x(x+2d)} \tag{3.4}$$

so that

$$\xi = \int_0^x \left(\frac{\beta^2 (x+d+1)^2 + \alpha^2}{x(x+2d)}\right)^{1/2} dx, \tag{3.5}$$

then (3.3) can be written as

$$\frac{d^2 G_l}{d\xi^2} = \left(K^2 - \frac{1}{4\xi^2} + g(\xi)\right) G_l, \tag{3.6}$$

where

$$g(\xi) = \left(\frac{\xi'''}{2\xi'^3} - \frac{3\xi''^2}{4\xi'^4} - \frac{d^2}{x^2(x+2d)^2}\frac{1}{\xi'^2}\right) + \frac{1}{4\xi^2}. \tag{3.7}$$

By substituting for ξ', ξ'', and ξ''' in terms of x, (3.7) can be written in the form

$$h(x) + \frac{1}{4\xi^2} - \frac{[\beta^2 (d+1)^2 + \alpha^2]^2 d^2}{4[\beta^2 (x+d+1)^2 + \alpha^2]^3 (x+2d) x}, \tag{3.8}$$

where $h(x)$ is a positive, bounded function of x. By examining (3.8), $g(\xi)$ can be shown to be a slowly varying, bounded, function which is $O(1)$ as $\xi \to 0$ and $O(1/\xi^2)$ as $\xi \to \infty$. Furthermore the bound on the function $h(x)$ contains a factor $(x+2d)^{-1}$ which behaves like d^{-1} in the limit $x \to 0$, and therefore the bound of $h(x)$ diverges as $d \to 0$. Using (3.5), the remaining terms in (3.8) can be shown to be bounded with respect to d for *all* x. Since we wish to neglect $g(\xi)$ in comparison with K^2, d cannot take small values. This puts restrictions on a and Q, requiring that both are small compared with M, but provided this requirement is satisfied $g(\xi)$ can be neglected in (3.6). The resulting equation has exact solutions so that the approximate solutions of (3.6) are therefore given by

$$G_l(\xi) \simeq \begin{cases} \xi^{1/2} I_0(K\xi), \\ \xi^{1/2} K_0(K\xi), \end{cases} \tag{3.9}$$

where I_0 and K_0 are the modified Bessel functions. Using the transformations relating G_l to Z_l and Z_l to R_l, and (3.4), we have

$$R_l(x) \simeq \frac{1}{[x(x+2d)]^{1/2}} \frac{[x(x+2d)]^{1/4}}{[\beta^2(x+d+1)^2+\alpha^2]^{1/4}} \xi^{1/2} \begin{cases} I_0(K\xi), \\ K_0(K\xi), \end{cases} \tag{3.10}$$

from which we may define the two solutions

$$R_l^{(1)}(x) = \frac{\xi^{1/2} I_0(K\xi)}{[x(x+2d)]^{1/4}[\beta^2(x+d+1)^2+\alpha^2]^{1/4}} \tag{3.11}$$

and

$$R_l^{(2)}(x) = \frac{\xi^{1/2} K_0(K\xi)}{[x(x+2d)]^{1/4}[\beta^2(x+d+1)^2+\alpha^2]^{1/4}}. \tag{3.12}$$

4. Approximate form for the potential

Using the solutions (3.11) and (3.12) an approximate solution to (2.9) can be obtained, as in I. Imposing continuity of $R_l(x)$ at $x = x_b$ and integrating (2.9) across the δ-function gives

$$\left.\frac{\mathrm{d}R_l}{\mathrm{d}x}\right|_{x_b+0} - \left.\frac{\mathrm{d}R_l}{\mathrm{d}x}\right|_{x_b-0} = g\frac{S_l(0)}{[x_b(x_b+2d)]^{1/2}[M^2(x_b+d+1)^2+a^2]^{1/2}}. \tag{4.1}$$

Combining (4.1) with the physical requirement that the solution should be bounded as $x \to 0$ and tend to zero as $x \to \infty$ we get

$$R_l(x) \simeq \begin{cases} -g\dfrac{[x_b(x_b+2d)]^{1/2} S_l(0)}{[M^2(x_b+d+1)^2+a^2]^{1/2}} R_l^{(2)}(x_b) R_l^{(1)}(x), & 0 \leqslant x \leqslant x_b, \\ -g\dfrac{[x_b(x_b+2d)]^{1/2} S_l(0)}{[M^2(x_b+d+1)^2+a^2]^{1/2}} R_l^{(1)}(x_b) R_l^{(2)}(x), & x_b \leqslant x < \infty. \end{cases} \tag{4.2}$$

Finally the full solution of (2.3) can be obtained by substituting (4.2) into (2.4) to give

$$\Phi(x,\theta) \simeq \begin{cases} -g\displaystyle\sum_{l=0}^{\infty} \frac{[x_b(x_b+2d)]^{1/2}}{[M^2(x_b+d+1)^2+a^2]^{1/2}} S_l(0) R_l^{(2)}(x_b) R_l^{(1)}(x) S_l(\theta), & 0 \leqslant x \leqslant x_b, \\ -g\displaystyle\sum_{l=0}^{\infty} \frac{[x_b(x_b+2d)]^{1/2}}{[M^2(x_b+d+1)^2+a^2]^{1/2}} S_l(0) R_l^{(1)}(x_b) R_l^{(2)}(x) S_l(\theta), & x_b \leqslant x < \infty, \end{cases} \tag{4.3}$$

which written explicitly for $x_b \leqslant x < \infty$, and putting $\xi(x_b) = \xi_b$,

$$\Phi(x, \theta) \simeq -g \sum_{l=0}^{\infty} \frac{[x_b(x_b+2d)]^{1/2}}{[M^2(x_b+d+1)^2+a^2]^{1/2}} \times \frac{S_l(0)(K\xi_b)^{1/2} I_0(K\xi_b)(K\xi)^{1/2} K_0(K\xi) S_l(\theta)}{[x_b(x_b+2d)]^{1/4}[N^2(x_b+d+1)^2+\lambda_l^2]^{1/4}[x(x+2d)]^{1/4}[N^2(x+d+1)^2+\lambda_l^2]^{1/4}}. \tag{4.4}$$

The difference in appearance between this equation and (4.5) in I is due to the choice of normalisation for the spheroidal harmonics $S_l(\theta)$. It can be seen from (4.4) that provided the series is uniformly convergent (as proved to be the case for the equivalent equation in I) then $\Phi \to 0$ as $x_b \to 0$ and the dominant term in the fall-off is

$$\frac{[x_b(x_b+2d)]^{1/2}}{[M^2(x_b+d+1)^2+a^2]^{1/2}} \propto \frac{(b-r_+)^{1/2}(b-r_-)^{1/2}}{(b^2+a^2)^{1/2}}, \qquad (\text{as } b \to r_+). \tag{4.5}$$

In the Schwarzschild limit (4.5) has the same form as that found in I.

5. Discussion

The above result, in accordance with the 'no hair' theorem, shows that if a baryon is slowly lowered into a Kerr–Newman black hole along the axis of symmetry, then the field due to the baryon, outside the black hole, vanishes as the baryon approaches the event horizon. It is interesting to note that if the cylindrical symmetry of this problem is not made use of at the outset then there are difficulties at the horizon due to the singular nature of the coordinate system. Moreover these difficulties mean that the more general problem of a baryon located off the axis of symmetry cannot be handled by the method used above. To tackle this problem it is necessary to use a coordinate system which is well-behaved at the horizon (see, for example, Carter 1968).

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